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From the Birkhoff–Gustavson normalization to the Bertrand–Darboux integrability condition

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Abstract. The Bertrand–Darboux integrability condition (BDIC) for a certain class of perturbed harmonic oscillators is studied from the viewpoint of the Birkhoff–Gustavson (BG) normalization: in solving an inverse problem of the BG normalization in computer algebra, it is shown that if the perturbed harmonic oscillators with a homogeneous *cubic*-polynomial potential and with a homogeneous *quartic*-polynomial potential share the same BG-normal form up to degree four, then both oscillators satisfy the BDIC.

1. Introduction

The Bertrand–Darboux (BD) theorem is well known to provide a necessary and sufficient condition for two-degree-of-freedom natural Hamiltonian systems associated with the Euclidean metric to admit an integral of motion quadratic in momenta (Darboux 1901). Moreover, that condition is necessary and sufficient for the natural Hamiltonian systems to be separable in either Cartesian, polar, parabolic or elliptic coordinates (Marshall and Wojciechowski 1988). This theorem thereby provides a sufficient condition for the complete integrability of the natural Hamiltonian systems, which will be referred to as the BD integrability condition (BDIC) in this paper. The BDIC has been studied repeatedly from various viewpoints; the separation of variables (Marshall and Wojciechowski 1988, Grosche *et al* 1995 in the path-integral formulation), the complete integrability (Perelomov 1990), and the so-called direct method (Hietarinta 1987), for example (see also Whittaker 1937 as an older reference and the references in the above-cited literature).

The aim of this paper is to show that the BDIC is obtained as an outcome of a study on the Birkhoff–Gustavson (BG) normalization (Gustavson 1966, Moser 1968) for the Hamiltonians of the perturbed harmonic oscillators with homogeneous *cubic*-polynomial potentials (PHOCPs). In solving an *inverse problem* of the BG normalization for a given PHOCP Hamiltonian with a help of computer algebra, the family of the perturbed harmonic oscillators with homogeneous *quartic*-polynomial potentials (PHOQPs) is identified, which share the same BG-normal form up to degree four with the given PHOCP. Consequently, a new deep relation is found between the BDIC for the PHOCPs and that for the PHOQPs: it is shown that if a PHOCP and a PHOQP share the same BG-normal form up to degree four then both oscillators are integrable in the sense that they satisfy the BDIC. It is worth noting that the present work was inspired in the debugging process of the computer program named 'ANFER' for the BG normalization (Uwano *et al* 1999, Uwano 2000), where the one-parameter Hénon–Heiles system was taken as an example.

Before the outline of this paper, the inverse problem of the BG normalization is explained very briefly. The BG normalization has been a powerful method for nonlinear Hamiltonian systems. For example, when a two-degree-of-freedom Hamiltonian system with a 1 : 1 resonant equilibrium point is given, the BG normalization of its Hamiltonian around the equilibrium point provides an 'approximate' Hamiltonian system: the truncation of the normalized Hamiltonian up to a finite degree is associated with the approximate Hamiltonian system, which provides a good account of the surface of section with sufficiently small energies (Kummer 1976, Cushman 1982). Such a good approximation implies that finding the class of Hamiltonian systems admitting the same BG normalization up to a finite degree amounts to finding a class of Hamiltonian systems which admit the surface of section similar to each other. The following question has been hence posed by the author as an *inverse problem* of the BG normalization (Chekanov *et al* 1998, 2000, Uwano *et al* 1999): *what kind of polynomial Hamiltonian can be brought into a given polynomial Hamiltonian in BG-normal form?* Since elementary algebraic operations, differentiation, and integration of polynomials have to be repeated many times to solve the inverse problem, computer algebra is worth applying to solve the inverse problem (see Uwano *et al* 1999, Uwano 2000 for the program named 'ANFER' and Chekanov *et al* 1998, 2000 for 'GITA^{-1}').

The organization of this paper is outlined as follows. Section 2 sets up the ordinary and the inverse problems of the BG normalization for Hamiltonians, which will be often referred to as 'the ordinary problem for Hamiltonians' and 'the inverse problem for BG-normal form Hamiltonians', respectively, henceforth. Although the setting up of the ordinary problem seems to be merely a review of Moser (1968), it is of great use to define the ordinary problem in a mathematically sound form: through the review, the class of canonical transformation to be utilized in the BG normalization can be specified explicitly. In section 3, the one-parameter Hénon–Heiles Hamiltonian is taken as an example to illustrate how the ordinary and the inverse problems proceed. From the solution of the inverse problem, it follows that if the one-parameter Hénon–Heiles system and the PHOQP share the same BG-normal form up to degree four then both dynamical systems satisfy the BDIC, so they are integrable. In section 4, the discussion in section 3 made for the one-parameter Hénon–Heiles Hamiltonian is extended to the Hamiltonians of the PHOCPs: the ordinary and the inverse problems of the BG normalization for the PHOCP Hamiltonians are dealt with there. In the ordinary problem, the Hamiltonians for PHOCPs are normalized up to degree four. For the BG-normal form thus obtained, the inverse problem is solved up to degree four. Both of the problems are solved with (a prototype of) the symbolic-computing program ANFER working on reduce 3.6 (Uwano 2000). It is shown that if a PHOCP and a PHOQP share the same BG-normal form up to degree four then both oscillators are integrable in the sense that they satisfy the BDIC. Section 5 is for concluding remarks including a conjecture on a further extension.

2. Setting up the ordinary and the inverse problems of BG normalization

In this section, the ordinary and the inverse problems of the BG normalization are reviewed for the two-degree-of-freedom Hamiltonians (Uwano *et al* 1999).

2.1. The ordinary problem

We start by defining the ordinary problem along with a review of Moser (1968). Let $\boldsymbol{R}^2 \times \boldsymbol{R}^2$ be the phase space with the canonical coordinates (q, p) $(q, p \in \boldsymbol{R}^2)$, and $K(q, p)$ be the

Hamiltonian expressed in power-series form,

$$K(q,p) = \tfrac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + \sum_{k=3}^{\infty} K_k(q,p) \tag{1}$$

where each $K_k(q,p)$ $(k = 3, 4, \ldots)$ is a homogeneous polynomial of degree k in (q,p).

Remark 1. The convergent radius of the power series (1) may vanish. This happens to any K that is not analytic but differentiable around the origin, for example. In such a case, the power series (1) is considered only in a formal sense. We will, however, often eliminate the word 'formal' from such formal power series henceforth.

We normalize the Hamiltonian K through the local canonical transformation of the following form. Let (ξ,η) be other canonical coordinates working around the origin of $\boldsymbol{R}^2 \times \boldsymbol{R}^2$. We consider the canonical transformation of (q,p) to (ξ,η) associated with a generating function (Goldstein 1950) in power-series form,

$$W(q,\eta) = \sum_{j=1}^{2} q_j\eta_j + \sum_{k=3}^{\infty} W_k(q,\eta) \tag{2}$$

where each $W_k(q,\eta)$ $(k = 3, 4, \ldots)$ is a homogeneous polynomial of degree k in (q,η). $W(q,\eta)$ is said to be of the second type since $W(q,\eta)$ is a function of the 'old' position variables q and the 'new' momentum ones η (see Goldstein 1950). The canonical transformation associated with $W(q,\eta)$ is given by the relation

$$(q,p) \to (\xi,\eta) \qquad \text{with} \quad p = \frac{\partial W}{\partial q} \qquad \text{and} \qquad \xi = \frac{\partial W}{\partial \eta} \tag{3}$$

which leaves the origin of $\boldsymbol{R}^2 \times \boldsymbol{R}^2$ invariant on account of (2). By $G(\xi,\eta)$, we denote the Hamiltonian brought from $K(q,p)$ by

$$G\left(\frac{\partial W}{\partial \eta}, \eta\right) = K\left(q, \frac{\partial W}{\partial q}\right) \tag{4}$$

through the transformation (3). The BG normalization for K is accomplished by choosing the generating function $W(q,\eta)$ in (4) to put $G(\xi,\eta)$ in BG-normal form.

Definition 2.1. *Let $G(\xi,\eta)$ be written in the power-series form,*

$$G(\xi,\eta) = \tfrac{1}{2}\sum_{j=1}^{2}(\eta_j^2 + \xi_j^2) + \sum_{k=3}^{\infty} G_k(\xi,\eta) \tag{5}$$

where each $G_k(\xi,\eta)$ $(k = 3, 4, \ldots)$ is a homogeneous polynomial of degree k in (ξ,η). The power series $G(\xi,\eta)$ is said to be in BG-normal form up to degree r if and only if

$$\left\{\tfrac{1}{2}\sum_{j=1}^{2}(\eta_j^2 + \xi_j^2), G_k(\xi,\eta)\right\}_{\xi,\eta} = 0 \qquad (k = 3, \ldots, r) \tag{6}$$

holds true, where $\{\cdot,\cdot\}_{\xi,\eta}$ is the canonical Poisson bracket (Arnold 1980) in (ξ,η).

Let us equate the homogeneous-polynomial part of degree k $(k = 3, 4, \ldots)$ in (4). Then equation (4) is put into the series of equations,

$$G_k(q,\eta) + (D_{q,\eta}W_k) = K_k(q,\eta) + \Phi_k(q,\eta) \qquad (k = 3, 4, \ldots) \tag{7}$$

where $D_{q,\eta}$ is the differential operator,

$$D_{q,\eta} = \sum_{j=1}^{2}\left(q_j\frac{\partial}{\partial \eta_j} - \eta_j\frac{\partial}{\partial q_j}\right). \tag{8}$$

The $\Phi_k(q,\eta)$ in (7) is the homogeneous polynomial of degree k in (q,η) which is uniquely determined by the $W_3,\ldots,W_{k-1}, K_3,\ldots,K_{k-1}, G_3,\ldots,G_{k-1}$ given: in particular, we have $\Phi_3(q,\eta)=0$ and

$$\Phi_4(q,\eta)=\sum_{j=1}^{2}\left(\frac{1}{2}\left(\frac{\partial W_3}{\partial q_j}\right)^2+\left.\frac{\partial K_3}{\partial p_j}\right|_{(q,\eta)}\frac{\partial W_3}{\partial q_j}-\frac{1}{2}\left(\frac{\partial W_3}{\partial \eta_j}\right)^2-\left.\frac{\partial G_3}{\partial \xi_j}\right|_{(q,\eta)}\frac{\partial W_3}{\partial \eta_j}\right). \tag{9}$$

Since we will only deal with the BG normalization up to degree four in this paper, we will not obtain the expression of Φ_k for $k>4$ in more detail (see Uwano *et al* 1999, if necessary).

To solve equation (7), the direct-sum decomposition induced by $D_{q,\eta}$ of the spaces of homogeneous polynomials is of great use. Let $V_k(q,\eta)$ denote the vector space of homogeneous polynomials of degree k in (q,η) with real-valued coefficients ($k=0,1,\ldots$). Since the differential operator $D_{q,\eta}$ acts linearly on each $V_k(q,\eta)$, the action of $D_{q,\eta}$ naturally induces the direct-sum decomposition,

$$V_k(q,\eta)=\text{image } D^{(k)}_{q,\eta}\oplus\ker D^{(k)}_{q,\eta}\qquad(k=0,1,\ldots) \tag{10}$$

of $V_k(q,\eta)$, where $D^{(k)}_{q,\eta}$ denotes the restriction,

$$D^{(k)}_{q,\eta}=D_{q,\eta}|_{V_k(q,\eta)}\qquad(k=0,1,\ldots). \tag{11}$$

Remark 2. We have $\ker D^{(k)}_{q,\eta}=\{0\}$ and image $D^{(k)}_{q,\eta}=V_k(q,\eta)$ if k is odd.

For $D_{q,\eta}$ and $D^{(k)}_{q,\eta}$ ($k=3,4,\ldots$), we have the following easily shown lemma.

Lemma 2.2. *Equation (6) is equivalent to*

$$(D_{q,\eta}(G_k|_{(q,\eta)}))(q,\eta)=(D^{(k)}_{q,\eta}(G_k|_{(q,\eta)}))(q,\eta)=0\qquad(k=3,\ldots,r). \tag{12}$$

Namely, $G_k|_{(q,\eta)}\in\ker D^{(k)}_{q,\eta}$ *(*$k=3,\ldots,r$*).*

According to (10), let us decompose $K_k(q,\eta)$ and $\Phi_k(q,\eta)$ ($k=3,4,\ldots$) to be

$$\begin{aligned}K_k(q,\eta)&=K_k^{\text{image}}(q,\eta)+K_k^{\text{ker}}(q,\eta)\\ \Phi_k(q,\eta)&=\Phi_k^{\text{image}}(q,\eta)+\Phi_k^{\text{ker}}(q,\eta)\end{aligned} \tag{13}$$

where

$$\begin{aligned}&K_k^{\text{image}}(q,\eta),\ \Phi_k^{\text{image}}(q,\eta)\in\text{image } D^{(k)}_{q,\eta}\\ &K_k^{\text{ker}}(q,\eta),\ \Phi_k^{\text{ker}}(q,\eta)\in\ker D^{(k)}_{q,\eta}.\end{aligned} \tag{14}$$

Since $G_k(q,\eta)\in\ker D^{(k)}_{q,\eta}$ by lemma 2.2 and since $D_{q,\eta}W_k\in\text{image } D^{(k)}_{q,\eta}$, we obtain

$$G_k(q,\eta)=K_k^{\text{ker}}(q,\eta)+\Phi_k^{\text{ker}}(q,\eta)\qquad(k=3,4,\ldots) \tag{15}$$

as a solution of (7), where W_k is chosen to be

$$W_k(q,\eta)=(D^{(k)}_{q,\eta}|^{-1}_{\text{image } D^{(k)}_{q,\eta}}(K_k^{\text{image}}|_{(q,\eta)}+\Phi_k^{\text{image}}|_{(q,\eta)}))(q,\eta). \tag{16}$$

What is crucial in (16) is that $W_k\in\text{image } D^{(k)}_{q,\eta}$ ($k=3,4,\ldots$): for a certain integer $\kappa\geqslant 3$, let us consider the sum, $\tilde{W}_\kappa=W_\kappa+(\text{any polynomial in }\ker D^{(\kappa)}_{q,\eta})$, where G_k and W_k with $k<\kappa$ are given by (15) and (16). Even after such a modification, $\tilde{W}_\kappa$ satisfy (7) with $k=\kappa$ still, which leads to another series of solutions of (7) with $k>\kappa$. Therefore, under the restriction $W_k\in\text{image } D^{(k)}_{q,\eta}$ ($k=3,4,\ldots$), we can say that (15) with (16) is the unique solution of (4). To summarize, the ordinary problem is defined as follows.

Definition 2.3 (The ordinary problem). *For a given Hamiltonian $K(q, p)$ in power series (1), bring $K(q, p)$ into the BG-normal form $G(\xi, \eta)$ in power series (5) which satisfy (4) and (6) with $r = \infty$, where the second-type generating function $W(q, \eta)$ of the form (2) is chosen to satisfy (4) and*

$$W_k(q, \eta) \in \text{image } D^{(k)}_{q,\eta} \qquad (k = 3, 4, \ldots). \tag{17}$$

Theorem 2.4. *The BG-normal form $G(\xi, \eta)$ for the Hamiltonian $K(q, p)$ is given by (5) with (15), where the second-type generating function $W(q, \eta)$ in (4) is chosen to be (2) subject to (16).*

Remark 3. The convergent radius of the BG-normal form G in power series (5) vanishes in general (see Moser 1968). In such a case, G is considered only in a formal sense. However, as in remark 1 for K, we will often eliminate 'formal' from such formal power series henceforth.

2.2. The inverse problem

To define the inverse problem of the BG normalization appropriately, we review the key equation (4) of the ordinary problem from a viewpoint of canonical transformations. Let us regard the power series $W(q, \eta)$ in (4) as a third-type generating function, a generating function of the 'new' position variables q and the 'old' momentum ones η (Goldstein 1950), which provides the inverse canonical transformation,

$$(\xi, \eta) \to (q, p) \qquad \text{with} \quad \xi = -\frac{\partial(-W)}{\partial \eta} \qquad \text{and} \qquad p = -\frac{\partial(-W)}{\partial q} \tag{18}$$

of (3). Equation (4) is rewritten as

$$K\left(q, -\frac{\partial(-W)}{\partial q}\right) = G\left(-\frac{\partial(-W)}{\partial \eta}, \eta\right) \tag{19}$$

which is combined with (18) to show the following.

Lemma 2.5. *Let $G(\xi, \eta)$ of (5) be the BG-normal form for the Hamiltonian $K(q, p)$ of (1), which satisfies (4) with a second-type generating function $W(q, \eta) \in$ image $D_{q,\eta}$. The Hamiltonian $K(q, p)$ is restored from $G(\xi, \eta)$ through the canonical transformation (18) associated with the third-type generating function $-W(q, \eta) \in$ image $D_{q,\eta}$.*

We are now in a position to pose the inverse problem in the following way: let the Hamiltonian $H(q, p)$ be written in the form

$$H(q, p) = \tfrac{1}{2}\sum_{j=1}^{n}(p_j^2 + q_j^2) + \sum_{k=3}^{\infty} H_k(q, p) \tag{20}$$

where each $H_k(q, p)$ $(k = 3, 4, \ldots)$ is a homogeneous polynomial of degree k in (q, p). Further, let a third-type generating function $S(q, \eta)$ be written in the form

$$S(q, \eta) = -\sum_{j=1}^{n} q_j \eta_j - \sum_{k=3}^{\infty} S_k(q, \eta) \tag{21}$$

where each $S_k(q, \eta)$ $(k = 3, 4, \ldots)$ is a homogeneous polynomial of degree k in (q, η).

Definition 2.6 (The inverse problem). *For a given BG-normal form, $G(\xi, \eta)$, in power series (5), identify all the Hamiltonians $H(q, p)$ in power series (20) which satisfy*

$$H\left(q, -\frac{\partial S}{\partial q}\right) = G\left(-\frac{\partial S}{\partial \eta}, \eta\right) \tag{22}$$

where the third-type generating function $S(q, \eta)$ in power series (21) is chosen to satisfy (22) and

$$S_k(q, \eta) \in \text{image } D_{q,\eta}^{(k)} \qquad (k = 3, 4, \ldots). \tag{23}$$

We solve the inverse problem in the following way. On equating the homogeneous-polynomial part of degree k in (22), equation (22) is put into the series of equations,

$$H_k(q, \eta) - (D_{q,\eta} S_k)(q, \eta) = G_k(q, \eta) - \Psi_k(q, \eta) \qquad (k = 3, 4, \ldots) \tag{24}$$

where $D_{q,\eta}$ is given by (8). The $\Psi_k(q, \eta)$ is the homogeneous polynomial of degree k in (q, η) determined uniquely by the $H_3, \ldots, H_{k-1}, G_3, \ldots, G_{k-1}, S_3, \ldots, S_{k-1}$ given. In particular, we have $\Psi_3(q, \eta) = 0$ and

$$\Psi_4(q, \eta) = \sum_{j=1}^{2} \left(\frac{1}{2} \left(\frac{\partial S_3}{\partial q_j} \right)^2 + \left. \frac{\partial H_3}{\partial p_j} \right|_{(q,\eta)} \frac{\partial S_3}{\partial q_j} - \frac{1}{2} \left(\frac{\partial S_3}{\partial \eta_j} \right)^2 - \left. \frac{\partial G_3}{\partial \xi_j} \right|_{(q,\eta)} \frac{\partial S_3}{\partial \eta_j} \right). \tag{25}$$

Remark 4. As is easily seen from (9) and (25), Ψ_4 takes a similar form to Φ_4 due to (21). Indeed, since the substitution $W = -S$ in (4) provides (22), Ψ_k for each $k > 4$ can be obtained as Φ_k given by (7) with $H_3, \ldots, H_{k-1}$ and $S_3, \ldots, S_{k-1}$ in place of $K_3, \ldots, K_{k-1}$ and $W_3, \ldots, W_{k-1}$. Such a similarity will be utilized effectively in future in writing the program ANFER for the ordinary and the inverse problems in a unified form.

As in the ordinary problem, we solve (24) for H_k and S_k by using the direct-sum decomposition (10) of $V_k(q, \eta)$, the vector spaces of homogeneous polynomials of degree k ($k = 0, 1, \ldots$). Let us decompose H_k and Ψ_k ($k = 3, 4, \ldots$) to be

$$\begin{aligned} H_k(q, \eta) &= H_k^{\text{image}}(q, \eta) + H_k^{\text{ker}}(q, \eta) \\ \Psi_k(q, \eta) &= \Psi_k^{\text{image}}(q, \eta) + \Psi_k^{\text{ker}}(q, \eta) \end{aligned} \tag{26}$$

where

$$\begin{aligned} &H_k^{\text{image}}(q, \eta), \Psi_k^{\text{image}}(q, \eta) \in \text{image } D_{q,\eta}^{(k)} \\ &H_k^{\text{ker}}(q, \eta), \Psi_k^{\text{ker}}(q, \eta) \in \ker D_{q,\eta}^{(k)}. \end{aligned} \tag{27}$$

Then on equating the $\ker D_{q,\eta}^{(k)}$ part in (24), H_k^{ker} is determined to be

$$H_k^{\text{ker}}(q, \eta) = G_k(q, \eta) - \Psi_k^{\text{ker}}(q, \eta). \tag{28}$$

Equating the image $D_{q,\eta}$ part of (24), we have

$$H_k^{\text{image}}(q, \eta) - (D_{q,\eta} S_k)(q, \eta) = -\Psi_k^{\text{image}}(q, \eta) \qquad (k = 3, 4, \ldots). \tag{29}$$

Since the pair of unidentified polynomials H_k^{image} and S_k exists in (29), H_k^{image} is not determined uniquely in contrast with H_k^{ker}; such a non-uniqueness is of the very nature of the inverse problem. Accordingly, equation (29) is solved as

$$H_k^{\text{image}}(q, \eta) \in \text{image } D_{q,\eta}^{(k)} : \text{chosen arbitrarily} \tag{30}$$

$$S_k(q, \eta) = (D_{q,\eta}^{(k)}|_{\text{image } D_{q,\eta}^{(k)}}^{-1} (H_k^{\text{image}}|_{(q,\eta)} + \Psi_k^{\text{image}}|_{(q,\eta)}))(q, \eta) \tag{31}$$

($k = 3, 4, \ldots$). Now we have the following.

Theorem 2.7. *For a given BG-normal form $G(\xi, \eta)$ in power series (5), the solution $H(q, p)$ of the inverse problem is given by (20) subject to (28) and (30), where the third-type generating function $S(q, \eta)$ in (22) is chosen to be (21) subject to (31).*

2.3. The degree-2δ ordinary and inverse problems

In the preceding subsections, we have defined the ordinary and the inverse problems of BG normalization, and then have found their solutions in power series. From a practical point of view, however, we usually deal with the BG-normal forms not in power series but in polynomials. Indeed, as mentioned in section 1, when we utilize the BG normalization to provide an approximate system for a given system, we truncate the normalized Hamiltonian up to a finite degree. Hence it is natural to think of a 'finite-degree version' of both the ordinary and the inverse problems (Uwano *et al* 1999).

Definition 2.8 (The degree-2δ ordinary problem). *For a given Hamiltonian $K(q, p)$ of the form (1) (possibly in polynomial form) and an integer $\delta \geqslant 2$, bring K into the polynomial $G(\xi, \eta)$ of degree 2δ in BG-normal form which satisfies (4) up to degree 2δ, where the second-type generating function $W(q, \eta)$ in (4) is chosen to be the polynomial of degree 2δ subject to (4) up to degree 2δ and (17) with $k = 3, \ldots, 2\delta$.*

Remark 5. The reason why we think of only the even- (2δ-) degree case is that the BG-normal form of any 1 : 1 resonant Hamiltonian consists of even-degree terms only (see remark 2 and lemma 2.2).

Definition 2.9 (The degree-2δ inverse problem). *For a given BG-normal form, $G(\xi, \eta)$, of degree 2δ with an integer $\delta \geqslant 2$, identify all the polynomial Hamiltonians $H(q, p)$ of degree 2δ which satisfy (22) up to degree 2δ, where the third-type generating function $S(q, \eta)$ is chosen to be the polynomial of degree 2δ subject to (22) up to degree 2δ and (23) with $k = 3, \ldots, 2\delta$.*

In closing this section, we wish to mention the way to solve the (degree-2δ) ordinary and the inverse problems in computer algebra. Although the discussion throughout this section is mathematically complete, it is not easy to calculate even in computer algebra (15), (16) and (28), (30), (31) as they are presented because we are faced with a highly combinatorial difficulty in calculating Φ_k and Ψ_k: to calculate Ψ_k for example, $G_3, \ldots, G_{k-1}, S_3, \ldots, S_{k-1}$ and $H_3, \ldots, H_{k-1}$ have to be kept on computer. What is worse is that the dimension, $\sum_{h=1}^{4} \binom{4}{h}\binom{\ell-1}{h-1}$, of V_ℓ to which G_ℓ, S_ℓ and H_ℓ belong rises in a combinatorial manner as ℓ increases, where the symbol $\binom{\cdot}{\cdot}$ indicates the binomial coefficient. These facts will cause a combinatorial increase of the memory size in the computer required for calculation. To get rid of such a difficulty, we break the transformations, (4) and (22) into a recursion of certain canonical transformations of simpler form, which will be presented in the appendix for the degree-four case.

3. Example: the one-parameter Hénon–Heiles system

In this section, we take the one-parameter Hénon–Heiles Hamiltonian

$$K_\mu(q, p) = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + q_1^2 q_2 + \mu q_2^3 \qquad (\mu \in \boldsymbol{R}) \tag{32}$$

as an example to illustrate how the ordinary and the inverse problems proceed. As mentioned in section 1, this example inspired the present work.

3.1. The degree-four ordinary and inverse problems

In order to present the results in compact forms, the complex variables defined by

$$z_j = q_j + \mathrm{i}p_j \qquad \zeta_j = \xi_j + \mathrm{i}\eta_j \qquad (j = 1, 2) \tag{33}$$

will be of great use. In terms of z and $\bar{z}$, K_μ is written in the form

$$\begin{aligned}K_\mu(q,p) = {}& \tfrac{1}{2}(z_1\bar{z}_1 + z_2\bar{z}_2) + \tfrac{1}{8}\mu(z_2^3 + 3z_2^2\bar{z}_2 + 3z_2\bar{z}_2^2 + \bar{z}_2^3) \\ & + \tfrac{1}{8}(z_1^2 z_2 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2 + \bar{z}_1^2\bar{z}_2 + 2z_1\bar{z}_1 z_2 + 2z_1\bar{z}_1\bar{z}_2). \end{aligned} \tag{34}$$

Using a prototype of ANFER (Uwano *et al* 1999, Uwano 2000), we see that the BG-normal form for K_μ is given, up to degree four, by

$$\begin{aligned}G_\mu(\xi,\eta) = {}& \tfrac{1}{2}(\zeta_1\bar{\zeta}_1 + \zeta_2\bar{\zeta}_2) + \tfrac{1}{48}\{-5\zeta_1^2\bar{\zeta}_1^2 - 45\mu^2\zeta_2^2\bar{\zeta}_2^2 - (8+36\mu)\zeta_1\zeta_2\bar{\zeta}_1\bar{\zeta}_2 \\ & + 3\mu\zeta_1^2\bar{\zeta}_2^2 + 3\mu\zeta_2^2\bar{\zeta}_1^2 - 6\zeta_1^2\bar{\zeta}_2^2 - 6\zeta_2^2\bar{\zeta}_1^2\} \end{aligned} \tag{35}$$

which is well known (see Kummer 1976, Cushman 1982). We proceed to the degree-four inverse problem for the BG-normal form G_μ in turn. Solving (22) with G_μ in place of G by ANFER, we have the Hamiltonians in the polynomial of degree four of the following form as the solution:

$$H_\mu(q,p) = \tfrac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + H_{\mu,3}(q,p) + H_{\mu,4}(q,p) \tag{36}$$

with

$$\begin{aligned}H_{\mu,3}(q,p) = {}& a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2 + a_4 z_2^3 + a_5 z_1^2\bar{z}_1 \\ & + a_6 z_1^2\bar{z}_2 + a_7 z_1 z_2\bar{z}_1 + a_8 z_1 z_2\bar{z}_2 + a_9 z_2^2\bar{z}_1 + a_{10} z_2^2\bar{z}_2 \\ & + \bar{a}_1\bar{z}_1^3 + \bar{a}_2\bar{z}_1^2\bar{z}_2 + \bar{a}_3\bar{z}_1\bar{z}_2^2 + \bar{a}_4\bar{z}_2^3 + \bar{a}_5 z_1\bar{z}_1^2 \\ & + \bar{a}_6 z_2\bar{z}_1^2 + \bar{a}_7 z_1\bar{z}_1\bar{z}_2 + \bar{a}_8 z_2\bar{z}_1\bar{z}_2 + \bar{a}_9 z_1\bar{z}_2^2 + \bar{a}_{10} z_2\bar{z}_2^2 \end{aligned} \tag{37}$$

and

$$\begin{aligned}H_{\mu,4}(q,p) = {}& c_1 z_1^4 + c_2 z_1^3 z_2 + c_3 z_1^2 z_2^2 + c_4 z_1 z_2^3 + c_5 z_2^4 + c_6 z_1^3\bar{z}_1 + c_7 z_1^3\bar{z}_2 \\ & + c_8 z_1^2 z_2\bar{z}_1 + c_9 z_1^2 z_2\bar{z}_2 + c_{10} z_1 z_2^2\bar{z}_1 + c_{11} z_1 z_2^2\bar{z}_2 + c_{12} z_2^3\bar{z}_1 + c_{13} z_2^3\bar{z}_2 \\ & + \bar{c}_1\bar{z}_1^4 + \bar{c}_2\bar{z}_1^3\bar{z}_2 + \bar{c}_3\bar{z}_1^2\bar{z}_2^2 + \bar{c}_4\bar{z}_1\bar{z}_2^3 + \bar{c}_5\bar{z}_2^4 + \bar{c}_6 z_1\bar{z}_1^3 + \bar{c}_7 z_2\bar{z}_1^3 \\ & + \bar{c}_8 z_1\bar{z}_1^2\bar{z}_2 + \bar{c}_9 z_2\bar{z}_1^2\bar{z}_2 + \bar{c}_{10} z_1\bar{z}_1\bar{z}_2^2 + \bar{c}_{11} z_2\bar{z}_1\bar{z}_2^2 + \bar{c}_{12} z_1\bar{z}_2^3 + \bar{c}_{13} z_2\bar{z}_2^3 \\ & + 8(a_9\bar{a}_{10} z_2^2\bar{z}_1\bar{z}_2 + a_9\bar{a}_9 z_1 z_2\bar{z}_1\bar{z}_2 + a_{10}\bar{a}_9 z_1 z_2\bar{z}_2^2 \\ & + a_5\bar{a}_6 z_1 z_2\bar{z}_1^2 + a_6\bar{a}_5 z_1^2\bar{z}_1\bar{z}_2 + a_6\bar{a}_6 z_1 z_2\bar{z}_1\bar{z}_2) \\ & + 6(a_1\bar{a}_1 z_1^2\bar{z}_1^2 + a_{10}\bar{a}_{10} z_2^2\bar{z}_2^2 + a_4\bar{a}_4 z_2^2\bar{z}_2^2 + a_5\bar{a}_5 z_1^2\bar{z}_1^2) \\ & + 4(a_8\bar{a}_5 z_1 z_2\bar{z}_1\bar{z}_2 + a_8\bar{a}_6 z_2^2\bar{z}_1\bar{z}_2 + a_8\bar{a}_9 z_1^2\bar{z}_2^2 + a_9\bar{a}_7 z_1 z_2\bar{z}_1^2 \\ & + a_9\bar{a}_8 z_2^2\bar{z}_1^2 + a_1\bar{a}_2 z_1^2\bar{z}_1\bar{z}_2 + a_{10}\bar{a}_7 z_1 z_2\bar{z}_1\bar{z}_2 + a_2\bar{a}_1 z_1 z_2\bar{z}_1^2 \\ & + a_3\bar{a}_4 z_1 z_2\bar{z}_2^2 + a_4\bar{a}_3 z_2^2\bar{z}_1\bar{z}_2 + a_5\bar{a}_8 z_1 z_2\bar{z}_1\bar{z}_2 + a_6\bar{a}_7 z_1^2\bar{z}_2^2 \\ & + a_6\bar{a}_8 z_1 z_2\bar{z}_2^2 + a_7\bar{a}_{10} z_1 z_2\bar{z}_1\bar{z}_2 + a_7\bar{a}_6 z_2^2\bar{z}_1^2 + a_7\bar{a}_9 z_1^2\bar{z}_1\bar{z}_2) \\ & + \tfrac{8}{3}(a_2\bar{a}_2 z_1 z_2\bar{z}_1\bar{z}_2 + a_3\bar{a}_3 z_1 z_2\bar{z}_1\bar{z}_2) \\ & + 2(a_8\bar{a}_{10} z_1 z_2\bar{z}_2^2 + a_8\bar{a}_7 z_1^2\bar{z}_1\bar{z}_2 + a_8\bar{a}_7 z_1 z_2\bar{z}_2^2 + a_8\bar{a}_8 z_2^2\bar{z}_2^2 \\ & + a_1\bar{a}_3 z_1^2\bar{z}_2^2 + a_{10}\bar{a}_8 z_2^2\bar{z}_1\bar{z}_2 + a_2\bar{a}_4 z_1^2\bar{z}_2^2 + a_3\bar{a}_1 z_2^2\bar{z}_1^2 \\ & + a_4\bar{a}_2 z_2^2\bar{z}_1^2 + a_5\bar{a}_7 z_1^2\bar{z}_1\bar{z}_2 + a_7\bar{a}_5 z_1 z_2\bar{z}_1^2 + a_7\bar{a}_7 z_1^2\bar{z}_1^2 \\ & + a_7\bar{a}_8 z_1 z_2\bar{z}_1^2 + a_7\bar{a}_8 z_2^2\bar{z}_1\bar{z}_2 - a_8\bar{a}_6 z_1 z_2\bar{z}_1^2 - a_7\bar{a}_9 z_1 z_2\bar{z}_2^2 \\ & - a_9\bar{a}_5 z_2^2\bar{z}_1^2 - a_9\bar{a}_7 z_2^2\bar{z}_1\bar{z}_2 - a_9\bar{a}_9 z_2^2\bar{z}_2^2 - a_{10}\bar{a}_6 z_2^2\bar{z}_1^2 \\ & - a_5\bar{a}_9 z_1^2\bar{z}_2^2 - a_6\bar{a}_{10} z_1^2\bar{z}_2^2 - a_6\bar{a}_6 z_1^2\bar{z}_1^2 - a_6\bar{a}_8 z_1^2\bar{z}_1\bar{z}_2) \\ & + \tfrac{4}{3}(a_2\bar{a}_3 z_1^2\bar{z}_1\bar{z}_2 + a_2\bar{a}_3 z_1 z_2\bar{z}_2^2 + a_3\bar{a}_2 z_1 z_2\bar{z}_1^2 + a_3\bar{a}_2 z_2^2\bar{z}_1\bar{z}_2) \\ & + \tfrac{2}{3}(a_2\bar{a}_2 z_1^2\bar{z}_1^2 + a_3\bar{a}_3 z_2^2\bar{z}_2^2) + \tfrac{1}{48}(-8z_1 z_2\bar{z}_1\bar{z}_2 - 5z_1^2\bar{z}_1^2 - 6z_1^2\bar{z}_2^2 - 6z_2^2\bar{z}_1^2 \\ & - 36\mu z_1 z_2\bar{z}_1\bar{z}_2 - 45\mu^2 z_2^2\bar{z}_2^2 + 3\mu z_1^2\bar{z}_2^2 + 3\mu z_2^2\bar{z}_1^2) \end{aligned} \tag{38}$$

where a_h ($h = 1, \ldots, 10$) and c_ℓ ($\ell = 1, \ldots, 13$) are complex-valued parameters chosen arbitrarily. In (38), the polynomial with the coefficients (c_ℓ) expresses $H_{\mu,4}^{\text{image}}$ and the polynomial whose coefficients are written in terms of (a_h) and (f_k) expresses $H_{\mu,4}^{\ker}$. From the lengthy expression (36)–(38), one might understand an effectiveness of computer algebra in the inverse problem.

It is worth pointing out that if we choose (a_h) and (c_ℓ) to be

$$\begin{aligned} &a_1 = a_3 = a_5 = a_8 = a_9 = 0 \\ &2a_2 = 2a_6 = a_7 = \tfrac{1}{4} \qquad 3a_4 = a_{10} = \frac{3\mu}{8} \end{aligned} \tag{39}$$

and

$$c_\ell = 0 \qquad (\ell = 1, \ldots, 13) \tag{40}$$

respectively, H_μ becomes equal to the one-parameter Hénon–Heiles Hamiltonian K_μ.

3.2. The BDIC

We wish to find the condition for (a_h) and (c_ℓ) to bring H_μ into the Hamiltonian of the PHOQP. To bring H_μ into a PHOQP Hamiltonian, we have to make $H_{\mu,3}$ vanish. The vanishing of $H_{\mu,3}$ is realized by the substitution

$$a_h = 0 \qquad (h = 1, \ldots, 10) \tag{41}$$

in (37). We bring $H_{\mu,4}|_{a=0}(q, p)$ into a homogeneous polynomial of degree four in q in turn. To do this, we have to find the set of non-vanishing $((c_\ell), (\lambda_m))$ for which the following identities for z hold true:

$$\begin{aligned} \lambda_1 q_1^4 &= c_1 z_1^4 + c_6 z_1^3 \bar{z}_1 - \tfrac{5}{48} z_1^2 \bar{z}_1^2 + \bar{c}_1 \bar{z}_1^4 + \bar{c}_6 z_1 \bar{z}_1^3 \\ \lambda_2 q_1^3 q_2 &= c_2 z_1^3 z_2 + + c_7 z_1^3 \bar{z}_2 + c_8 z_1^2 z_2 \bar{z}_1 + \bar{c}_2 \bar{z}_1^3 \bar{z}_2 + \bar{c}_7 z_2 \bar{z}_1^3 + \bar{c}_8 z_1 \bar{z}_1^2 \bar{z}_2 \\ \lambda_3 q_1 q_2^3 &= c_{11} z_1 z_2^2 \bar{z}_2 + c_{12} z_2^3 \bar{z}_1 + c_4 z_1 z_2^3 + \bar{c}_4 \bar{z}_1 \bar{z}_2^3 + \bar{c}_{12} z_1 \bar{z}_2^3 + \bar{c}_{11} z_2 \bar{z}_1 \bar{z}_2^2 \\ \lambda_4 q_2^4 &= c_5 z_2^4 + c_{13} z_2^3 \bar{z}_2 - \tfrac{15}{16}\mu^2 z_2^2 \bar{z}_2^2 + \bar{c}_{13} z_2 \bar{z}_2^3 + \bar{c}_5 \bar{z}_2^4 \end{aligned} \tag{42}$$

$$\begin{aligned} \lambda_5 q_1^2 q_2^2 = {} & c_{10} z_1 z_2^2 \bar{z}_1 + c_3 z_1^2 z_2^2 + c_9 z_1^2 z_2 \bar{z}_2 + \bar{c}_{10} z_1 \bar{z}_1 \bar{z}_2^2 + \bar{c}_3 \bar{z}_1^2 \bar{z}_2^2 + \bar{c}_9 z_2 \bar{z}_1^2 \bar{z}_2 \\ & + \tfrac{1}{16}\mu z_1^2 \bar{z}_2^2 + \tfrac{1}{16}\mu z_2^2 \bar{z}_1^2 - \tfrac{3}{4}\mu z_1 z_2 \bar{z}_1 \bar{z}_2 - \tfrac{1}{6} z_1 z_2 \bar{z}_1 \bar{z}_2 - \tfrac{1}{8} z_1^2 \bar{z}_2^2 - \tfrac{1}{8} z_2^2 \bar{z}_1^2 \end{aligned} \tag{43}$$

where λ_m ($m = 1, \ldots, 5$) are real-valued parameters, and $q_j = (z_j + \bar{z}_j)/2$ ($j = 1, 2$). To make the identities hold true from the first to the fourth, we have to choose the parameters appearing in those identities to be

$$\begin{aligned} &\lambda_1 = 16c_1 = 4c_6 = -\tfrac{5}{18} \qquad \lambda_2 = c_2 = c_7 = c_8 = 0 \\ &\lambda_3 = c_4 = c_{11} = c_{12} = 0 \qquad \lambda_4 = 16c_5 = 4c_{13} = -\frac{5\mu^2}{2}. \end{aligned} \tag{44}$$

The fifth identity holds true if and only if the overdetermined system of equations,

$$\begin{aligned} &\lambda_5 = 16c_3 = 8c_9 = 8c_{10} \\ &\lambda_5 = -3\mu - \tfrac{2}{3} \\ &\lambda_5 = \mu - 2 \end{aligned} \tag{45}$$

admits a solution. By a simple calculation, we see that (45) admits the solution

$$\lambda_5 = 16c_3 = 8c_9 = 8c_{10} = -\tfrac{5}{3} \tag{46}$$

if and only if μ satisfies

$$\mu = \tfrac{1}{3}. \tag{47}$$

To summarize, we have the following.

Theorem 3.1. *The one-parameter Hénon–Heiles Hamiltonian $K_\mu(q, p)$ shares its BG-normal form up to degree four with the Hamiltonian of the PHOQP if and only if the parameter μ satisfies (47). Under (47), the PHOQP Hamiltonian sharing the BG-normal form with $K_{1/3}$ is given by*

$$Q(q, p) = \tfrac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) - \tfrac{5}{18}(q_1^4 + 6q_1^2q_2^2 + q_2^4). \tag{48}$$

After theorem 3.1, one might ask whether $K_{1/3}$ and Q have specific meanings or not. Those who are familiar with the separability of dynamical systems may recognize immediately that $H_{1/3}$ and Q are separable in $q_1 \pm q_2$ (see Perelomov 1990). We can hence answer this question affirmatively owing to the BD theorem concerning not only the separability but also the integrability. The theorem is stated as follows (see Marshall and Wojciechowski 1988, Yamaguchi and Nambu 1998, Grosche *et al* 1995, for example).

Theorem 3.2 (Bertrand–Darboux). *Let F be a natural Hamiltonian of the form*

$$F(q, p) = \tfrac{1}{2}\sum_{j=1}^{2} p_j^2 + V(q) \tag{49}$$

where $V(q)$ is a differentiable function in q. Then, the following three statements are equivalent for the Hamiltonian system with F.

(1) There exists a set of real-valued constants, $(\alpha, \beta, \beta', \gamma, \gamma') = (0, 0, 0, 0, 0)$, for which $V(q)$ satisfies

$$\begin{aligned}\left(\frac{\partial^2 V}{\partial q_2^2} - \frac{\partial^2 V}{\partial q_1^2}\right)&(-2\alpha q_1 q_2 - \beta' q_2 - \beta q_1 + \gamma)\\ &+2\frac{\partial^2 V}{\partial q_1 \partial q_2}(\alpha q_2^2 - \gamma q_1^2 + \beta q_2 - \beta' q_1 + \gamma')\\ &+\frac{\partial V}{\partial q_1}(6\alpha q_2 + 3\beta) - \frac{\partial V}{\partial q_2}(6\alpha q_1 + 3\beta') = 0.\end{aligned} \tag{50}$$

(2) The Hamiltonian system with F admits an integral of motion quadratic in momenta.

(3) The Hamiltonian F is separable in either Cartesian, polar, parabolic or elliptic coordinates.

Due to the statement (2) in theorem 3.2, a natural Hamiltonian system with F is always integrable if (50) holds true. In this regard, we will refer to (50) as the BDIC henceforth.

For the PHOCPs and the PHOQPs, Yamaguchi and Nambu (1998) have given a more explicit expression of the BDIC (50) convenient for our purpose:

Lemma 3.3. *Let $\mathcal{F}^{(k)}(q, p)$ $(k = 3, 4)$ be the Hamiltonians of the form*

$$\mathcal{F}^{(k)}(q, p) = \tfrac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + \mathcal{P}^{(k)}(q) \qquad (k = 3, 4) \tag{51}$$

with

$$\mathcal{P}^{(3)}(q) = f_1 q_1^3 + f_2 q_1^2 q_2 + f_3 q_1 q_2^2 + f_4 q_2^3 \tag{52}$$

or

$$\mathcal{P}^{(4)}(q) = g_1 q_1^4 + g_2 q_1^3 q_2 + g_3 q_1^2 q_2^2 + g_4 q_1 q_2^3 + g_5 q_2^4 \tag{53}$$

where f_k ($k = 1, \ldots, 4$) and g_ℓ ($\ell = 1, \ldots, 5$) are real-valued parameters. For the PHOCP with $\mathcal{F}^{(3)}(q, p)$, the BDIC (50) is equivalent to one of the following conditions: (54a), (54b), or (54c);

$$3(f_1 f_3 + f_2 f_4) - (f_2^2 + f_3^2) = 0 \tag{54a}$$

$$f_1 = 2f_3 \qquad f_2 = f_4 = 0 \tag{54b}$$

$$f_4 = 2f_2 \qquad f_1 = f_3 = 0. \tag{54c}$$

For the PHOQP with $\mathcal{F}^{(4)}(q, p)$, the BDIC (50) is equivalent to one of the following conditions: (55a) or (55b);

$$\begin{aligned} &g_3 = 2g_1 = 2g_5 \\ &g_2 = g_4 = 0 \end{aligned} \tag{55a}$$

$$\begin{aligned} &9g_2^2 + 4g_3^2 - 24g_1 g_3 - 9g_2 g_4 = 0 \\ &9g_4^2 + 4g_3^2 - 24g_3 g_5 - 9g_2 g_4 = 0 \\ &(g_2 + g_4)g_3 - 6(g_1 g_4 + g_2 g_5) = 0. \end{aligned} \tag{55b}$$

The integrability of the Hénon–Heiles system with $\mu = \frac{1}{3}$ and the PHOQP with Q can be confirmed now by showing that (50) holds true both for $K_{1/3}$ and Q. Let us start with showing the integrability of the one-parameter Hénon–Heiles system with $\mu = \frac{1}{3}$. If we choose (f_h) to be

$$f_1 = f_3 = 0 \qquad f_2 = 1 \qquad f_4 = \tfrac{1}{3} \tag{56}$$

the Hamiltonian $\mathcal{F}^{(3)}$ becomes $K_{1/3}$. Evidently, the (f_h) given by (56) satisfy the BDIC (54*a*), so that the one-parameter Hénon–Heiles system with $\mu = \frac{1}{3}$ is integrable.

We proceed to showing the integrability of the PHOQP with Q given by (48) in turn. In order to bring $\mathcal{F}^{(4)}$ into Q, we choose (g_ℓ) to be

$$6g_1 = g_3 = 6g_5 = -\tfrac{5}{3} \qquad g_2 = g_4 = 0. \tag{57}$$

By a simple calculation, we see that the (g_ℓ) given by (57) satisfy the BDIC (55*b*), so the PHOQP with Q is integrable. To summarize, we have the following.

Theorem 3.4. *If the one-parameter Hénon–Heiles system and the perturbed harmonic oscillator with a homogeneous-quartic polynomial potential admit the same BG normalization up to degree four, then both dynamical systems are integrable in the sense that they satisfy the BDIC.*

4. Extension: the PHOCPs

4.1. The degree-four ordinary and the inverse problems of PHOCP

In this section, the discussion in section 3 for the one-parameter Hénon–Heiles system is extended to the PHOCPs. To start with, we solve the degree-four ordinary problem for the PHOCP Hamiltonian $\mathcal{F}^{(3)}(q, p)$ given by (51) with (52). By ANFER, we see that the BG-normal form for $\mathcal{F}^{(3)}$ is given, up to degree four, by

$$\begin{aligned} \mathcal{G}(\xi, \eta) = {} & \tfrac{1}{2}(\zeta_1\overline{\zeta}_1 + \zeta_2\overline{\zeta}_2) - \tfrac{15}{16}(f_1^2\zeta_1^2\overline{\zeta}_1^2 + f_4^2\zeta_2^2\overline{\zeta}_2^2) - \tfrac{3}{4}(f_1 f_3\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2 + f_2 f_4\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2) \\ & - \tfrac{5}{8}(f_1 f_2\zeta_1^2\overline{\zeta}_1\overline{\zeta}_2 + f_1 f_2\zeta_1\zeta_2\overline{\zeta}_1^2 + f_3 f_4\zeta_1\zeta_2\overline{\zeta}_2^2 + f_3 f_4\zeta_2^2\overline{\zeta}_1\overline{\zeta}_2) \\ & - \tfrac{5}{24}(f_2 f_3\zeta_1^2\overline{\zeta}_1\overline{\zeta}_2 + f_2 f_3\zeta_1\zeta_2\overline{\zeta}_1^2 + f_2 f_3\zeta_1\zeta_2\overline{\zeta}_2^2 + f_2 f_3\zeta_2^2\overline{\zeta}_1\overline{\zeta}_2) \\ & - \tfrac{1}{6}(f_2^2\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2 + f_3^2\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2) - \tfrac{5}{48}(f_2^2\zeta_1^2\overline{\zeta}_1^2 + f_3^2\zeta_2^2\overline{\zeta}_2^2) \end{aligned}$$

$$-\tfrac{1}{8}(f_2^2\zeta_1^2\overline{\zeta}_2^2+f_2^2\zeta_2^2\overline{\zeta}_1^2+f_3^2\zeta_1^2\overline{\zeta}_2^2+f_3^2\zeta_2^2\overline{\zeta}_1^2)$$
$$+\tfrac{1}{16}(f_1f_3\zeta_1^2\overline{\zeta}_2^2+f_1f_3\zeta_2^2\overline{\zeta}_1^2+f_2f_4\zeta_1^2\overline{\zeta}_2^2+f_2f_4\zeta_2^2\overline{\zeta}_1^2). \tag{58}$$

Note that if we choose (f_h) to be

$$f_1=f_3=0 \qquad f_2=1 \qquad f_4=\mu \tag{59}$$

$\mathcal{G}$ becomes G_μ, the BG-normal form for the one-parameter Hénon–Heiles Hamiltonian.

We solve the degree-four inverse problem for the BG-normal form $\mathcal{G}$ in turn: by ANFER, we have the following polynomial of degree four as the solution;

$$\mathcal{H}(q,p)=\tfrac{1}{2}\sum_{j=1}^{2}(p_j^2+q_j^2)+\mathcal{H}_3(q,p)+\mathcal{H}_4(q,p) \tag{60}$$

with

$$\begin{aligned}\mathcal{H}_3(q,p)={}&a_1z_1^3+a_2z_1^2z_2+a_3z_1z_2^2+a_4z_2^3+a_5z_1^2\overline{z}_1+a_6z_1^2\overline{z}_2\\&+a_7z_1z_2\overline{z}_1+a_8z_1z_2\overline{z}_2+a_9z_2^2\overline{z}_1+a_{10}z_2^2\overline{z}_2\\&+\overline{a}_1\overline{z}_1^3+\overline{a}_2\overline{z}_1^2\overline{z}_2+\overline{a}_3\overline{z}_1\overline{z}_2^2+\overline{a}_4\overline{z}_2^3+\overline{a}_5z_1\overline{z}_1^2+\overline{a}_6z_2\overline{z}_1^2\\&+\overline{a}_7z_1\overline{z}_1\overline{z}_2+\overline{a}_8z_2\overline{z}_1\overline{z}_2+\overline{a}_9z_1\overline{z}_2^2+\overline{a}_{10}z_2\overline{z}_2^2\end{aligned} \tag{61}$$

and

$$\begin{aligned}\mathcal{H}_4(q,p)={}&c_1z_1^4+c_2z_1^3z_2+c_3z_1^2z_2^2+c_4z_1z_2^3+c_5z_2^4+c_6z_1^3\overline{z}_1+c_7z_1^3\overline{z}_2+c_8z_1^2z_2\overline{z}_1\\&+c_9z_1^2z_2\overline{z}_2+c_{10}z_1z_2^2\overline{z}_1+c_{11}z_1z_2^2\overline{z}_2+c_{12}z_2^3\overline{z}_1+c_{13}z_2^3\overline{z}_2\\&+\overline{c}_1\overline{z}_1^4+\overline{c}_2\overline{z}_1^3\overline{z}_2+\overline{c}_3\overline{z}_1^2\overline{z}_2^2+\overline{c}_4\overline{z}_1\overline{z}_2^3+\overline{c}_5\overline{z}_2^4+\overline{c}_6z_1\overline{z}_1^3+\overline{c}_7z_2\overline{z}_1^3+\overline{c}_8z_1\overline{z}_1^2\overline{z}_2\\&+\overline{c}_9z_2\overline{z}_1^2\overline{z}_2+\overline{c}_{10}z_1\overline{z}_1\overline{z}_2^2+\overline{c}_{11}z_2\overline{z}_1\overline{z}_2^2+\overline{c}_{12}z_1\overline{z}_2^3+\overline{c}_{13}z_2\overline{z}_2^3\\&+8(a_6\overline{a}_6z_1z_2\overline{z}_1\overline{z}_2+a_9\overline{a}_9z_1z_2\overline{z}_1\overline{z}_2+a_5\overline{a}_6z_1z_2\overline{z}_1^2\\&+a_6\overline{a}_5z_1^2\overline{z}_1\overline{z}_2+a_9\overline{a}_{10}z_2^2\overline{z}_1\overline{z}_2+a_{10}\overline{a}_9z_1z_2\overline{z}_2^2)\\&+6(a_1\overline{a}_1z_1^2\overline{z}_1^2+a_4\overline{a}_4z_2^2\overline{z}_2^2+a_5\overline{a}_5z_1^2\overline{z}_1^2+a_{10}\overline{a}_{10}z_2^2\overline{z}_2^2)\\&+4(a_1\overline{a}_2z_1^2\overline{z}_1\overline{z}_2+a_8\overline{a}_9z_1^2\overline{z}_2^2+a_3\overline{a}_4z_1z_2\overline{z}_2^2+a_5\overline{a}_8z_1z_2\overline{z}_1\overline{z}_2\\&+a_6\overline{a}_7z_1^2\overline{z}_2^2+a_6\overline{a}_8z_1z_2\overline{z}_2^2+a_7\overline{a}_9z_1^2\overline{z}_1\overline{z}_2+a_7\overline{a}_{10}z_1z_2\overline{z}_1\overline{z}_2)\\&+4(a_2\overline{a}_1z_1z_2\overline{z}_1^2+a_4\overline{a}_3z_2^2\overline{z}_1\overline{z}_2+a_8\overline{a}_5z_1z_2\overline{z}_1\overline{z}_2+a_7\overline{a}_6z_2^2\overline{z}_1^2\\&+a_8\overline{a}_6z_2^2\overline{z}_1\overline{z}_2+a_9\overline{a}_7z_1z_2\overline{z}_1^2+a_{10}\overline{a}_7z_1z_2\overline{z}_1\overline{z}_2+a_9\overline{a}_8z_2^2\overline{z}_1^2)\\&+\tfrac{8}{3}(a_2\overline{a}_2z_1z_2\overline{z}_1\overline{z}_2+a_3\overline{a}_3z_1z_2\overline{z}_1\overline{z}_2)\\&+2(-a_6\overline{a}_6z_1^2\overline{z}_1^2+a_7\overline{a}_7z_1^2\overline{z}_1^2+a_8\overline{a}_8z_2^2\overline{z}_2^2-a_9\overline{a}_9z_2^2\overline{z}_2^2)\\&+2(a_1\overline{a}_3z_1^2\overline{z}_2^2+a_2\overline{a}_4z_1^2\overline{z}_2^2+a_5\overline{a}_7z_1^2\overline{z}_1\overline{z}_2-a_5\overline{a}_9z_1^2\overline{z}_2^2\\&-a_6\overline{a}_8z_1^2\overline{z}_1\overline{z}_2-a_6\overline{a}_{10}z_1^2\overline{z}_2^2+a_7\overline{a}_8z_1z_2\overline{z}_1^2\\&+a_7\overline{a}_8z_2^2\overline{z}_1\overline{z}_2-a_7\overline{a}_9z_1z_2\overline{z}_2^2+a_8\overline{a}_{10}z_1z_2\overline{z}_2^2)\\&+2(a_3\overline{a}_1z_2^2\overline{z}_1^2+a_4\overline{a}_2z_2^2\overline{z}_1^2+a_7\overline{a}_5z_1z_2\overline{z}_1^2-a_9\overline{a}_5z_2^2\overline{z}_1^2\\&-a_8\overline{a}_6z_1z_2\overline{z}_1^2-a_{10}\overline{a}_6z_2^2\overline{z}_1^2+a_8\overline{a}_7z_1^2\overline{z}_1\overline{z}_2\\&+a_8\overline{a}_7z_1z_2\overline{z}_2^2-a_9\overline{a}_7z_2^2\overline{z}_1\overline{z}_2+a_{10}\overline{a}_8z_2^2\overline{z}_1\overline{z}_2)\\&+\tfrac{4}{3}(a_2\overline{a}_3z_1^2\overline{z}_1\overline{z}_2+a_2\overline{a}_3z_1z_2\overline{z}_2^2+a_3\overline{a}_2z_1z_2\overline{z}_1^2+a_3\overline{a}_2z_2^2\overline{z}_1\overline{z}_2)\\&+\tfrac{2}{3}(a_2\overline{a}_2z_1^2\overline{z}_1^2+a_3\overline{a}_3z_2^2\overline{z}_2^2)\\&-\tfrac{15}{16}(f_1^2z_1^2\overline{z}_1^2+f_4^2z_2^2\overline{z}_2^2)-\tfrac{3}{4}(f_1f_3z_1z_2\overline{z}_1\overline{z}_2+f_2f_4z_1z_2\overline{z}_1\overline{z}_2)\\&-\tfrac{5}{8}(f_1f_2z_1^2\overline{z}_1\overline{z}_2+f_1f_2z_1z_2\overline{z}_1^2+f_3f_4z_1z_2\overline{z}_2^2+f_3f_4z_2^2\overline{z}_1\overline{z}_2)\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{5}{24} f_2 f_3(z_1^2\bar{z}_1\bar{z}_2 + z_1 z_2\bar{z}_1^2 + z_1 z_2\bar{z}_2^2 + z_2^2\bar{z}_1\bar{z}_2) - \tfrac{1}{6}(f_2^2 z_1 z_2\bar{z}_1\bar{z}_2 + f_3^2 z_1 z_2\bar{z}_1\bar{z}_2)\\
&-\tfrac{1}{8}(f_2^2 z_1^2\bar{z}_2^2 + f_2^2 z_2^2\bar{z}_1^2 + f_3^2 z_1^2\bar{z}_2^2 + f_3^2 z_2^2\bar{z}_1^2) - \tfrac{5}{48}(f_2^2 z_1^2\bar{z}_1^2 + f_3^2 z_2^2\bar{z}_2^2)\\
&+\tfrac{1}{16}(f_1 f_3 z_2^2\bar{z}_1^2 + f_1 f_3 z_1^2\bar{z}_2^2 + f_2 f_4 z_1^2\bar{z}_2^2 + f_2 f_4 z_2^2\bar{z}_1^2)
\end{aligned}
\tag{62}
$$

where a_h $(h = 1, \ldots, 10)$ and c_ℓ $(\ell = 1, \ldots, 13)$ are the complex-valued parameters chosen arbitrarily, and f_k $(k = 1, \ldots, 4)$ the real-valued parameters in $\mathcal{P}^{(3)}(q)$ (see (52)). Namely, we have 46 degrees of freedom in the solution, $\mathcal{H}$, of the inverse problem of the PHOCP with (f_k) fixed. Note that if (f_k) are chosen to be (59) then $\mathcal{H}$ becomes H_μ given by (36)–(38). Further, if (a_h), (c_ℓ) and (f_k) are chosen to be (39), (40) and (59), respectively, then $\mathcal{H}$ becomes K_μ given by (32). After (60)–(62), one might understand more than after (36)–(38) the necessity of computer algebra in the inverse problem.

4.2. The BDIC

We wish to find the condition for (a_h), (c_ℓ) and (f_k) to bring $\mathcal{H}$ into the PHOQP Hamiltonian $\mathcal{F}^{(4)}(q, p)$ defined by (51) with (53). In order to make $\mathcal{H}_3(q, p)$ vanish (see (61)), we have to choose (a_h) to be (41). Our next task is then to bring $\mathcal{H}_4|_{a=0}(q, p)$ into a certain $\mathcal{P}^{(4)}(q)$. Let ν_n $(n = 1, \ldots, 5)$ be real-valued parameters. Then we see that $\mathcal{H}_4|_{a=0}$ takes the form $\mathcal{P}^{(4)}(q)$ if and only if the following identities of z hold true for a non-vanishing $((c_h), (\nu_n))$:

$$
\begin{aligned}
\nu_1 q_1^4 &= c_1 z_1^4 + c_6 z_1^3\bar{z}_1 - \tfrac{5}{48}(9f_1^2 + f_2^2)z_1^2\bar{z}_1^2 + \bar{c}_6 z_1\bar{z}_1^3 + \bar{c}_1\bar{z}_1^4\\
\nu_2 q_2^4 &= c_5 z_2^4 + c_{13} z_2^3\bar{z}_2 - \tfrac{5}{48}(9f_4^2 + f_3^2)z_2^2\bar{z}_2^2 + \bar{c}_{13} z_2\bar{z}_2^3 + \bar{c}_5\bar{z}_1^4\\
\nu_3 q_1^3 q_2 &= c_2 z_1^3 z_2 + c_7 z_1^3\bar{z}_2 + c_8 z_1^2 z_2\bar{z}_1 + \bar{c}_2\bar{z}_1^3\bar{z}_2 + \bar{c}_7\bar{z}_1^3 z_2 + \bar{c}_8\bar{z}_1^2\bar{z}_2 z_1\\
&\quad - \tfrac{5}{24}(3f_1 f_2 + f_2 f_3)(z_1^2\bar{z}_1\bar{z}_2 + z_1 z_2\bar{z}_1^2)\\
\nu_4 q_1 q_2^3 &= c_4 z_1 z_2^3 + c_{11} z_1 z_2^2\bar{z}_2 + c_{12} z_2^3\bar{z}_1 + \bar{c}_{12} z_1\bar{z}_2^3 + \bar{c}_{11} z_2\bar{z}_1\bar{z}_2^2 + \bar{c}_4\bar{z}_1\bar{z}_2^3\\
&\quad - \tfrac{5}{24}(3f_3 f_4 + f_2 f_3)(z_2^2\bar{z}_1\bar{z}_2 + z_1 z_2\bar{z}_2^2)\\
\nu_5 q_1^2 q_2^2 &= c_3 z_1^2 z_2^2 + c_9 z_1^2 z_2\bar{z}_2 + c_{10} z_1 z_2^2\bar{z}_1 + \bar{c}_{10} z_1\bar{z}_1\bar{z}_2^2 + \bar{c}_9 z_2\bar{z}_1^2\bar{z}_2 + \bar{c}_3\bar{z}_1^2\bar{z}_2^2\\
&\quad + \tfrac{1}{16}(f_1 f_3 + f_2 f_4 - 2f_2^2 - 2f_3^2)(z_1^2\bar{z}_2^2 + \bar{z}_1^2 z_2^2)\\
&\quad - \tfrac{1}{12}(9f_1 f_3 + 9f_2 f_4 + 2f_2^2 + 2f_3^2)z_1 z_2\bar{z}_1\bar{z}_2
\end{aligned}
\tag{63}
$$

where ν_n $(n = 1, \ldots, 5)$ are real-valued parameters and $q_j = (z_j + \bar{z}_j)/2$ $(j = 1, 2)$. As a necessary and sufficient condition for the identities to hold true from the first to the fourth, we have

$$
\begin{aligned}
\nu_1 &= 16c_1 = 4c_6 = -\tfrac{5}{18}(9f_1^2 + f_2^2)\\
\nu_2 &= 16c_5 = 4c_{13} = -\tfrac{5}{18}(9f_4^2 + f_3^2)\\
\nu_3 &= 16c_2 = 16c_7 = \tfrac{16}{3}c_8 = -\tfrac{10}{9}f_2(3f_1 + f_3)\\
\nu_4 &= 16c_4 = 16c_{12} = \tfrac{16}{3}c_{11} = -\tfrac{10}{9}f_3(3f_4 + f_2).
\end{aligned}
\tag{64}
$$

As for the fifth identity, we see that it holds true if and only if the following overdetermined system of equations:

$$
\begin{aligned}
\nu_5 &= 16c_3 = 8c_9 = 8c_{10}\\
\nu_5 &= (f_1 f_3 + f_2 f_4) - 2(f_2^2 + f_3^2)\\
\nu_5 &= -3(f_1 f_3 + f_2 f_4) - \tfrac{2}{3}(f_2^2 + f_3^2)
\end{aligned}
\tag{65}
$$

admits a solution. Surprisingly, the BDIC (54*a*) for PHOCPs comes out as a necessary and sufficient condition for (65) to admit a solution! Under (54*a*), equation (65) admits the solution

$$
\nu_5 = 16c_3 = 8c_9 = 8c_{10} = (f_1 f_3 + f_2 f_4) - 2(f_2^2 + f_3^2)
\tag{66}
$$

which is combined with (41) and (64) to bring $\mathcal{H}$ into the PHOQP Hamiltonian,

$$\mathcal{Q} = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) - \frac{5}{18}(9f_1^2 + f_2^2)q_1^4 - \frac{10}{9}(3f_1 + f_3)f_2q_1^3q_2 - \frac{5}{3}(f_2^2 + f_3^2)q_1^2q_2^2 - \frac{10}{9}(3f_4 + f_2)f_3q_1q_2^3 - \frac{5}{18}(9f_4^2 + f_3^2)q_2^4 \tag{67}$$

where (f_h) are subject to the BDIC (54a). We note here that the coefficient of $q_1^2q_2^2$ in (67) is obtained by combining (66) with (54a). To summarize, we have the following.

Theorem 4.1. *The perturbed harmonic-oscillator Hamiltonian $\mathcal{F}^{(3)}$ with a homogeneous cubic polynomial potential shares its BG-normal form with the perturbed harmonic-oscillator Hamiltonian $\mathcal{F}^{(4)}$ with a homogeneous-quartic potential up to degree four if and only if the PHOCP Hamiltonian $\mathcal{F}^{(3)}$ satisfies the BDIC (54a). Under (54a), the PHOQP Hamiltonian $\mathcal{F}^{(4)}$ sharing its BG-normal form with the PHOCP Hamiltonian $\mathcal{F}^{(3)}$ is equal to $\mathcal{Q}$ given by (67).*

We are now in a position to show the integrability of the PHOQP with $\mathcal{Q}$ subject to (54a). It is easy to see that $\mathcal{Q}$ is given by (51) with (53) under the substitution

$$\begin{aligned} g_1 &= -\tfrac{5}{18}(9f_1^2 + f_2^2) \qquad g_2 = -\tfrac{10}{9}(3f_1 + f_3)f_2 \qquad g_3 = -\tfrac{5}{3}(f_2^2 + f_3^2) \\ g_4 &= -\tfrac{10}{9}(3f_4 + f_2)f_3 \qquad g_5 = -\tfrac{5}{18}(9f_4^2 + f_3^2). \end{aligned} \tag{68}$$

A long but straightforward calculation shows that (g_ℓ) given by (68) with (54a) satisfy the BDIC (55b), so the PHOQP with $\mathcal{Q}$ is integrable.

Theorem 4.2. *If the PHOCP and a PHOQP share the same BG-normal form up to degree four, then both oscillators are integrable in the sense that they satisfy the BDIC.*

5. Concluding remarks

As shown in the previous section, the new deep relation between the BDIC for PHOCPs and that for PHOQPs is found (see theorems 4.1 and 4.2). The results obtained in this paper are expected to provide several interesting subjects, which are listed below.

(1) A further generalization of theorems 4.1 and 4.2 will be worth studying: a conjecture is posed as follows, which is investigated now.

Conjecture 5.1. *Let $\mathcal{F}^{(r)}(q,p)$ and $\mathcal{F}^{(2r-2)}(q,p)$ ($r = 3, 4, \ldots$) denote the perturbed harmonic oscillator Hamiltonians of the form*

$$\mathcal{F}^{(r)}(q,p) = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + V^{(r)}(q) \tag{69}$$

$$\mathcal{F}^{(2r-2)}(q,p) = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + V^{(2r-2)}(q) \tag{70}$$

where $V^{(2r-2)}$ is a homogeneous-polynomial potential of degree $(2r-2)$ and $V^{(r)}$ of degree r subject to

$$V^{(r)}(q) \in \text{image } D_{q,\eta}^{(r)}. \tag{71}$$

If the perturbed harmonic oscillators with the Hamiltonian $\mathcal{F}^{(r)}$ and with $\mathcal{F}^{(2r-2)}$ share the same BGNF up to degree-$(2r-2)$ then both oscillators are integrable in the sense that they satisfy the BDIC.

(2) As pointed out after theorem 3.1, the Hamiltonians $K_{1/3}$ and Q have a significant feature other than integrability: they are separable in $q_1 \pm q_2$. Such a separability can be found also in $\mathcal{F}^{(3)}$ with (54*a*) and $\mathcal{Q}$, the generalization of $K_{1/3}$ and Q, respectively. In fact, we can find, from theorem 4.1, the PHOCPs with

$$\mathcal{P}^{(3)}(q) = a(q_1 + q_2)^3 + b(q_1 - q_2)^3 \qquad (a, b \in \boldsymbol{R}) \tag{72}$$

as special cases of PHOCPs subject to the BDIC (54*a*), which cover all the PHOCP Hamiltonians separable in $q_1 \pm q_2$ (see Perelomov 1990). Further, theorem 4.1 implies that each of the separable PHOCPs (subject to (72)) shares the same BG-normal form with the PHOQP with

$$\mathcal{P}^{(4)}(q) = -5\{a^2(q_1 + q_2)^4 + b^2(q_1 - q_2)^4\} \tag{73}$$

which is also known to be separable in $q_1 \pm q_2$ (Perelomov 1990). It is worth noting that all the PHOQPs with (73) cover one-quarter of the class of all the PHOQP Hamiltonians separable in $q_1 \pm q_2$ in view of $a^2, b^2 \geqslant 0$. Thus we reach through theorem 4.1 the four-to-one correspondence between the PHOCPs with (72) and PHOQPs with (73) separable in $q_1 \pm q_2$. Since $\mathcal{F}^{(3)}$ with (54*a*) and $\mathcal{Q}$ are thought to include several classes of Hamiltonians separable in several coordinate systems other than $q_1 \pm q_2$, the separability will be worth studying extensively from the BG-normalization viewpoint in future.

(3) The perturbed oscillators referred to in theorems 4.1 and 4.2 are expected to provide good examples of the quantum bifurcation in the BG-normalized Hamiltonian systems (Uwano 1994, 1995, 1998, 1999): since the perturbed oscillators referred to in theorems 4.1 and 4.2 are integrable, their quantum spectra are expected to be obtained exactly. Then we will be able to decide whether or not the quantum bifurcation in the BG-normalized Hamiltonian system for those oscillators approximates the bifurcation in these oscillators to a good extent.

(4) In section 1, three approaches to the BDIC (or BD theorem) have been mentioned. As another approach to the BDIC, the work of Yamaguchi and Nambu (1998) is worth pointing out, in which the BDIC for the PHOCPs and the PHOQPs (see lemma 3.3) emerged from the renormalization of Hamiltonian equations. This will be an interesting problem in the future to study a relation between the BG normalization and the renormalization.

(5) As is easily seen, the solution (60)–(62) of the inverse problem for $\mathcal{G}$ admits 50 real-valued parameters. We may hence expect to obtain other integrable systems, the so-called electromagnetic type (Hietarinta 1987), for example.

On closing this section, we wish to mention of the role of computer algebra in this paper: without computer algebra, it would have been very difficult to find theorems 3.1, 3.4, 4.1 and 4.2.

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Appendix A. Mathematical basis for coding ANFER

In this appendix, we describe in mathematical terminology the algorithm of ANFER, a symbolic computing program, for the degree-four inverse problem on reduce 3.6 coded by the author. Although only a primitive prototype exists, those who are interested in ANFER will be able to see its source code at http://yang.amp.i.kyoto-u.ac.jp/~uwano/. After the algorithm, we present a key lemma supporting the algorithm. The general-degree case has been reported by Uwano *et al* (1999) briefly, and will be discussed in detail in a future paper by the author.

Appendix A.1. Setting up

The setting up of ANFER is performed as follows for the degree-2δ inverse problem. Let $\mathcal{S}^{(h)}(\xi^{(h)}, \eta^{(h-1)})$ $(h = 3, \ldots, 2\delta)$ be the third-type generating functions of the form

$$\mathcal{S}^{(h)}(\xi^{(h)}, \eta^{(h-1)}) = -\sum_{j=1}^{2} \xi^{(h)}\eta^{(h-1)} - \mathcal{S}_h(\xi^{(h)}, \eta^{(h-1)}) \qquad (h = 3, \ldots, 2\delta) \tag{A.1}$$

where each $\mathcal{S}_h$ is the homogeneous polynomial of degree h. With $\mathcal{S}^{(h)}$, we associate the canonical transformations,

$$\tau_h : (\xi^{(h-1)}, \eta^{(h-1)}) \to (\xi^{(h)}, \eta^{(h)}) \qquad (h = 3, \ldots, 2\delta) \tag{A.2}$$

with

$$\xi^{(h-1)} = -\frac{\partial \mathcal{S}^{(h)}}{\partial \eta^{(h-1)}} \qquad \eta^{(h)} = -\frac{\partial \mathcal{S}^{(h)}}{\partial \xi^{(h)}} \qquad (h = 3, \ldots, 2\delta). \tag{A.3}$$

From a given BG-normal form G of the form (5), we define the initial Hamiltonian $H^{(2)}$ to be the degree-2δ polynomial form,

$$H^{(2)}(\xi^{(2)}, \eta^{(2)}) = \tfrac{1}{2}\sum_{j=1}^{2}((\eta_j^{(2)})^2 + (\xi_j^{(2)})^2) + \sum_{k=3}^{2\delta} G_k(\xi^{(2)}, \eta^{(2)}). \tag{A.4}$$

In ANFER, the degree-2δ inverse problem,

$$H^{(h)}\left(\xi^{(h)}, -\frac{\partial \mathcal{S}^{(h)}}{\partial \xi^{(h)}}\right) = H^{(h-1)}\left(-\frac{\partial \mathcal{S}^{(h)}}{\partial \eta^{(h-1)}}, \eta^{(h-1)}\right) \tag{A.5}$$

$$\mathcal{S}^{(h)}(\xi^{(h)}, \eta^{(h-1)}) \in \text{image } D_{\xi^{(h)},\eta^{(h-1)}} \qquad \text{with} \quad \text{(A.1)} \tag{A.6}$$

for $H^{(h-1)}$ is solved recursively with $h = 3, \ldots, 2\delta$. The resultant $H^{(2\delta)}$ provides the solution H of the degree-2δ inverse problem for the BG-normal form G (Uwano *et al* 1999).

Appendix A.2. Algorithm for the degree-four inverse problem

We show how (A.5), (A.6) is solved in the degree-four case.

Step 1. Solving (A.5), (A.6) with $h = 3$.

Equating the homogeneous parts of degrees two, three and four in (A.5) with $h = 3$, we have

$$H_2^{(3)}(\xi^{(3)}, \eta^{(2)}) = H_2^{(2)}(\xi^{(3)}, \eta^{(2)}) \tag{A.7}$$

$$H_3^{(3)}(\xi^{(3)}, \eta^{(2)}) - (D_{\xi^{(3)},\eta^{(2)}}\mathcal{S}_3)(\xi^{(3)}, \eta^{(2)}) = H_3^{(2)}(\xi^{(3)}, \eta^{(2)}) \tag{A.8}$$

$$H_4^{(3)}(\xi^{(3)}, \eta^{(2)}) = H_4^{(2)}(\xi^{(3)}, \eta^{(2)}) - \sum_{j=1}^{2}\left(\frac{1}{2}\left(\frac{\partial \mathcal{S}_3}{\partial \xi_j^{(3)}}\right)^2 + \frac{\partial \mathcal{S}_3}{\partial \xi_j^{(3)}}\frac{\partial H_3^{(3)}}{\partial \eta_j^{(3)}}\right|_{(\xi^{(3)},\eta^{(2)})}$$

$$-\frac{1}{2}\left(\frac{\partial \mathcal{S}_3}{\partial \eta_j^{(2)}}\right)^2 - \frac{\partial \mathcal{S}_3}{\partial \eta_j^{(2)}} \frac{\partial H_3^{(2)}}{\partial \xi_j^{(2)}}\bigg|_{(\xi^{(3)},\eta^{(2)})}\Bigg). \tag{A.9}$$

In a similar way to (24)–(31), (A.8) is solved to be

$$\begin{aligned} &H_3^{(3)\,\mathrm{ker}}(\xi^{(3)}, \eta^{(2)}) = H_3^{(2)\,\mathrm{ker}}(\xi^{(3)}, \eta^{(2)}) = 0 \\ &(H_3^{(3)})^{\mathrm{image}}(\xi^{(3)}, \eta^{(2)}) \in \text{image } D^{(3)}_{\xi^{(3)},\eta^{(2)}} : \text{chosen arbitrarily} \\ &\mathcal{S}_3(\xi^{(3)}, \eta^{(2)}) = (D^{(3)}_{q,\eta}|^{-1}_{\text{image } D^{(3)}_{q,\eta}} H_3^{(3)\mathrm{image}}|_{(q,\eta)}))(\xi^{(3)}, \eta^{(2)}). \end{aligned} \tag{A.10}$$

$H_4^{(3)}$ is given by (A.9) with the substitution (A.10) into $H_3^{(3)}$ and $\mathcal{S}_3$.

Step 2. Solving (A.5), (A.6) with $h = 4$.

Equating the homogeneous parts of degrees two, three and four in (A.5) with $h = 4$, we have

$$H_k^{(4)}(\xi^{(4)}, \eta^{(3)}) = H_k^{(3)}(\xi^{(4)}, \eta^{(3)}) \qquad (k = 2, 3) \tag{A.11}$$

$$H_4^{(4)}(\xi^{(4)}, \eta^{(3)}) - (D_{\xi^{(4)},\eta^{(3)}}\mathcal{S}_4)(\xi^{(4)}, \eta^{(3)}) = H_4^{(3)}(\xi^{(4)}, \eta^{(3)}). \tag{A.12}$$

In a similar way to (24)–(31), (A.12) is solved to be

$$\begin{aligned} &H_4^{(4)\,\mathrm{ker}}(\xi^{(4)}, \eta^{(3)}) = H_3^{(4)\,\mathrm{ker}}(\xi^{(4)}, \eta^{(3)}) \\ &H_4^{(4)\mathrm{image}}(\xi^{(4)}, \eta^{(3)}) \in \text{image } D^{(4)}_{\xi^{(4)},\eta^{(3)}} : \text{chosen arbitrarily} \\ &\mathcal{S}_4(\xi^{(4)}, \eta^{(3)}) = (D^{(4)}_{q,\eta}|^{-1}_{\text{image } D^{(4)}_{q,\eta}} (H_4^{(4)\mathrm{image}}|_{(q,\eta)} - H_4^{(3)\mathrm{image}}|_{(q,\eta)}))(\xi^{(4)}, \eta^{(3)}). \end{aligned} \tag{A.13}$$

We are now in a position to show that $H^{(4)}(q, p)$ and $\mathcal{S}(q, \eta) = -\sum_{j=1}^{2} q_j\eta_j - \sum_{k=3}^{4} \mathcal{S}_k(q, \eta)$ are identical with $H(q, p)$ and $S(q, \eta)$, respectively, the solution of the degree-four inverse problem for G. Equation (25), $G_3(q, \eta) = 0$ and $\Psi_3(q, \eta) = 0$ (see the line above (25)) are put together with (28), (30) and (31) to show that $H_3(q, p)$ is equal to $H_3^{(3)}(q, p)$, so we have $S_3(q, \eta) = \mathcal{S}_3(q, \eta)$. Under $H_3(q, p) = H_3^{(3)}(q, p)$ and $S_3(q, \eta) = \mathcal{S}_3(q, \eta)$, we see that the second term in the rhs of (A.9) is equal to $-\Psi_4$ given by (25) with $(q, \eta) = (\xi^{(3)}, \eta^{(2)})$. This fact and (A.12) are put together with (A.13) to show that $H_4^{(4)}(q, p)$ and $\mathcal{S}_4(q, \eta)$ are identical with $H_4(q, p)$ and $S_4(q, \eta)$, respectively.

What is characteristic of the algorithm above is that the procedure for solving (24) up to $k = 4$ is divided into a pair of steps, steps 1 and 2 (extending to the degree-2δ case, we have $2\delta - 2$ steps). Although it is not so significant in the inverse problem of lower degree (like degree four), such a division will contribute a lot to the reduction of the memory size required in computation.

Appendix A.3. Composition of canonical transformations

We wish to give an explicit expression for the generating function associated with the composition $\tau_4 \circ \tau_3$ of canonical transformations, τ_3 and τ_4, which mathematically supports the algorithm given above. The general-degree version can be found in Uwano *et al* (1999) without a proof. Such an expression is really important because little is known explicitly of the composition of *non-infinitesimal* canonical transformations although that of infinitesimal ones is well known (Goldstein 1950).

Lemma A.1. *The composition $\tau_4 \circ \tau_3 : (\xi^{(2)}, \eta^{(2)}) \to (\xi^{(4)}, \eta^{(4)})$ of the canonical transformations τ_3 and τ_4 defined by (A.2) is associated with the third-type generating function*

$$\tilde{\mathcal{S}}(\xi^{(4)}, \eta^{(2)}) = \sum_{j=1}^{2} \tilde{\xi}_j \tilde{\eta}_j + \mathcal{S}^{(3)}(\tilde{\xi}, \eta^{(2)}) + \mathcal{S}^{(4)}(\xi^{(4)}, \tilde{\eta}) \tag{A.14}$$

where $\tilde{\xi}$ and $\tilde{\eta}$ are the functions $(\xi^{(4)}, \eta^{(2)})$ uniquely determined to satisfy

$$\tilde{\xi} = -\frac{\partial \mathcal{S}^{(4)}}{\partial \eta^{(3)}}(\xi^{(4)}, \tilde{\eta}) \qquad \tilde{\eta} = -\frac{\partial \mathcal{S}^{(3)}}{\partial \xi^{(3)}}(\tilde{\xi}, \eta^{(2)}) \tag{A.15}$$

around $(\xi^{(4)}, \eta^{(2)}) = 0$. $\tilde{\mathcal{S}}$ admits the expansion

$$\tilde{\mathcal{S}}(q, \eta) = -\sum_{j=1}^{2} q_j \eta_j - \mathcal{S}_3(q, \eta) - \mathcal{S}_4(q, \eta) + \mathrm{o}_4(q, \eta), \tag{A.16}$$

$$\frac{\mathrm{o}_4(q, \eta)}{\sqrt{\sum_{j=1}^{2}(q_j^2 + \eta_j^2)}^{\,4}} \to 0 \qquad \left(\sqrt{\sum_{j=1}^{2}(q_j^2 + \eta_j^2)} \to 0\right). \tag{A.17}$$

Proof. Since the third-type generating functions, $\mathcal{S}^{(h)}$s, satisfy

$$-\xi^{(h-1)}\,\mathrm{d}\eta^{(h-1)} - \eta^{(h)}\,\mathrm{d}\xi^{(h)} = \mathrm{d}\mathcal{S}^{(h)} \qquad (h = 3, 4) \tag{A.18}$$

(Goldstein 1950), we have

$$-\xi^{(2)}\,\mathrm{d}\eta^{(2)} - \eta^{(4)}\,\mathrm{d}\xi^{(4)} = \mathrm{d}\left\{\left(\sum_{j=1}^{2}\xi_j^{(3)}\eta_j^{(3)}\right) + \mathcal{S}^{(3)}(\xi^{(3)}, \eta^{(2)}) + \mathcal{S}^{(4)}(\xi^{(4)}, \eta^{(3)})\right\}. \tag{A.19}$$

On account of (A.3) with $h = 3, 4$, $\{(\xi^{(h)}, \eta^{(h)}\}_{h=2,3,4}$ are here restricted in the inverse image, $\tau^{-1}(0, 0, 0, 0)$, of the map,

$$\begin{aligned}&\tau : (\xi^{(2)}, \eta^{(2)}, \xi^{(3)}, \eta^{(3)}, \xi^{(4)}, \eta^{(4)}) \in \boldsymbol{R}^6 \\ &\mapsto \left(\xi^{(3)} + \frac{\partial \mathcal{S}^{(4)}}{\partial \eta^{(3)}}, \eta^{(3)} + \frac{\partial \mathcal{S}^{(3)}}{\partial \xi^{(3)}}, \xi^{(2)} + \frac{\partial \mathcal{S}^{(3)}}{\partial \eta^{(2)}}, \eta^{(4)} + \frac{\partial \mathcal{S}^{(4)}}{\partial \xi^{(4)}}\right) \in \boldsymbol{R}^4.\end{aligned} \tag{A.20}$$

Applying the implicit function theorem (Spivak 1965) to the map τ, we obtain uniquely the functions, $\tilde{\xi}$, $\tilde{\eta}$, $\hat{\xi}$ and $\hat{\eta}$, of $\xi^{(4)}, \eta^{(2)}$ subject to $\tau(\hat{\xi}, \eta^{(2)}, \tilde{\xi}, \tilde{\eta}, \hat{\eta}, \xi^{(4)}) = 0$ around $\xi^{(4)}, \eta^{(2)} = 0$. This shows (A.14) with (A.15). We move on to the proof of (A.16) with (A.17) in turn. Equation (A.15) is expanded in $(\xi^{(4)}, \eta^{(2)})$ as

$$\begin{aligned}\tilde{\xi} &= \xi^{(4)} + \frac{\partial \mathcal{S}_4}{\partial \eta^{(3)}}(\xi^{(4)}, \eta^{(2)}) + \mathrm{o}_3'(\xi^{(4)}, \eta^{(2)}) \\ \tilde{\eta} &= \eta^{(2)} + \frac{\partial \mathcal{S}_3}{\partial \xi^{(3)}}(\xi^{(4)}, \eta^{(2)}) + \mathrm{o}_3''(\xi^{(4)}, \eta^{(2)})\end{aligned} \tag{A.21}$$

where o_3' and o_3'' indicate the terms of order higher than three in $\sqrt{\sum_{j=1}^{2}(\xi_j^{(4)2} + \eta_j^{(2)2})}$. Substituting (A.21) into (A.14), we have (A.16). This completes the proof. □

We are now in a position to see why the algorithm given here works well. Lemma A.1 and the setting up (A.5), (A.6), are put together to imply that we have solved through steps 1 and 2 the degree-4 inverse problem

$$H^{(4)}\left(\xi^{(4)}, -\frac{\partial \tilde{\mathcal{S}}}{\partial \xi^{(4)}}\right) = H^{(2)}\left(-\frac{\partial \tilde{\mathcal{S}}}{\partial \eta^{(2)}}, \eta^{(2)}\right) \tag{A.22}$$

for $H^{(2)}$, where $\tilde{\mathcal{S}} \in \mathrm{image}D_{\xi^{(4)},\eta^{(2)}}$ up to degree four. This means that $\tilde{\mathcal{S}}(q, \eta)$ coincides up to degree four with the generating function $S(q, \eta)$ for the inverse problem for G, so that the degree-four inverse problem can be solved through steps 1 and 2.

References

Arnold V I 1980 *Mathematical Methods of Classical Mechanics* (New York: Springer) p 215
Chekanov N A, Hongo M, Rostovtsev V A, Uwano Y and Vinitsky S I 1998 *Phys. At. Nucl.* **61** 2029
Chekanov N A, Rostovtsev V A, Uwano Y and Vinitsky S I 2000 *Comput. Phys. Commun.* **126** 47
Cushman R 1982 *Proc. R. Soc.* A **382** 361
Darboux G 1901 *Arch. Neerlandaises* ii **6** 371
Goldstein H 1950 *Classical Mechanics* 2nd edn (Reading, MA: Addison-Wesley) p 241
Grosche C, Pogosyan G S and Sissakian A N 1995 *Fortschr. Phys.* **43** 453
Gustavson F G 1966 *Astron. J.* **71** 670
Hietarinta J 1987 *Phys. Rep.* **147** 87
Kummer M 1976 *Commun. Math. Phys.* **48** 53
Marshall I and Wojciechowski S 1988 *J. Math. Phys.* **29** 1338
Moser J K 1968 *Lectures on Hamiltonian Systems (Memoirs of American Mathematical Society 81)* (Providence, RI: American Mathematical Society) p 10
Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* vol 1 (Basel: Birkhäuser) p 81
Spivak M 1965 *Calculus on Manifolds* (New York: Benjamin) p 41
Uwano Y 1994 *J. Phys. Soc. Japan* Suppl. A **63** 31
——1995 *J. Phys. A: Math. Gen.* **28** 2041
——1998 *Int. J. Bifurcation and Chaos* **8** 941
——1999 *Rep. Math. Phys.* **44** 267
——2000 webpage http://yang.amp.i.kyoto-u.ac.jp/˜uwano/
Uwano Y, Chekanov N, Rostovtsev V and Vinitsky S 1999 *Computer Algebra in Scientific Computing* ed V G Ganzha *et al* (Berlin: Springer) p 441
Whittaker E T 1937 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* 4th edn (Cambridge: Cambridge University Press) p 332
Yamaguchi Y Y and Nambu Y 1998 *Prog. Theor. Phys.* **100** 199